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STABILITY OF THE TUBULAR LAYER OF A DEFORMED MATERIAL IN A ROTATING HORIZONTAL CYLINDER

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The stability conditions for the steady-state motion of the tubular layer of a treated deformable material in a rotating horizontal cylinder are determined analytically. With allowance for the accepted similarity criteria, universal diagrams of the boundaries of transition of modes of motion of liquid and loose materials in the cylinder are obtained on the basis of experimental data. Analysis of the diagrams shows the identity of the stability conditions for a liquid layer and a loose medium, which can be regarded as a Newtonian liquid upon fast relative motions. It is shown also that the analytical stability conditions for the liquid layer correspond to the experimental data for large Reynolds numbers when the mode hysteresis occurs and do not correspond to these data for small Reynolds numbers when secondary circulating flows form.

The problem of determining the modes of motion of a deformable material that partially fills a cylinder rotating around a horizontal axis is of great applied interest for specialists who study the dynamics of horizontal drum-type machines. The conditions of mutual transition of nontubular circulating and quasitubular solid-state forms of motion which correspond to low and high velocities of rotation are of great importance. Based on analysis of the working processes of these systems, one can reduce all forms of motion of a treated material to two most characteristic forms, namely, the motions of a viscous Newtonian liquid and a loose body.

In the mode of fast motion of a loose material, which can be regarded as a granular medium, its particles move chaotically, similarly to the molecules in a liquid, and when the flow propagates in the longitudinal direction, the particles begin to move in the transverse direction; these displacements transmit additional momenta from layer to layer and cause the appearance of viscous tangential stresses. The internal stresses in the medium arise owing to the momentum transfer and depend greatly on the shear velocity, as is the case in a liquid [1, 2]. The experimental data show that the flow of a loose material in a rotating horizontal cylinder is classified among fast motions; in this case, the material behaves as a viscous liquid [3–6]. Therefore, to describe the motion of a loose material in a cylinder, the mechanical model based on the Navier–Stokes equations is used.

The modes of motion of a liquid with low degree of filling of the cylinder hollow was studied experimentally by White and Higgins [3, 4]. In the case of the nontubular form of fluid motion, experimental and theoretical studies of the position of a free surface are given in [5] for high viscosity and in [6] for a high degree of filling. The problem of determination of high-speed modes of motion of a tubular layer of an ideal liquid was solved by Phillips [7]. The loss of motion stability of the surface of a viscous liquid which fills a considerable part of the hollow of a rotating cylinder was investigated in [8] with allowance for gravity. The problem of determination of the position of the free surface of the tubular layer of a viscous liquid with the use of the boundary-layer theory was solved analytically by Zhdan [9] and numerically by Deiber and Cerro [10]

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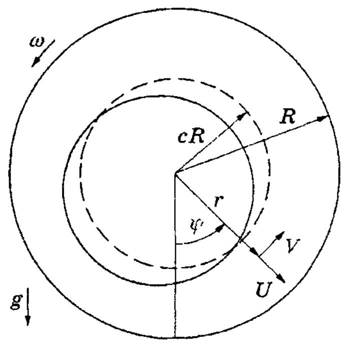

Fig. 1. Calculated flow pattern of the tubular layer of a deformable material.

with experimental check for a lamina. Badratinova [11] obtained the existence conditions for the plane flow regimes of the tubular layer of a viscous liquid. The nonexistence of these modes can cause three-dimensional secondary flows [12]. In [13], the characteristic modes of fluid motion are treated taking into account perturbations, and an attempt to generalize and extrapolate the results in the form of a two-parameter diagram is undertaken.

In the present study, the conditions of motion stability of the tubular layer of a deformable material treated as a viscous liquid in a cylinder are found analytically. The problem is formulated as in [7], but the liquid is assumed to be viscous. The motion similarity criteria are found and universal diagrams that determine the boundaries of transition of the characteristic modes of motion of a liquid and loose materials are obtained on the basis of experimental data.

We now consider a cylinder of radius R with smooth face walls partially filled with a liquid and rotating uniformly with angular velocity ω around the horizontal axis perpendicular to the acceleration of gravity g. With a large angular velocity of the cylinder, the liquid in the hollow takes the shape of a tubular layer with external radius R and free-surface radius cR $(0 \leqslant c \leqslant 1)$ (Fig. 1).

The fluid motion is considered in the plane perpendicular to the rotation axis of the cylinder. We introduce the polar coordinate system $(r,\ \varphi)$; U and V are the velocity components of the liquid. Then, the equations of motion and continuity have the form

$$\begin{gathered}
\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \varphi} - \frac{V^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g\cos\varphi + \nu\Big(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{2}{r^2}\frac{\partial V}{\partial \varphi} - \frac{U}{r^2}\Big), \\
\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \varphi} + \frac{UV}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} - g\sin\varphi + \nu\Big(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{2}{r^2}\frac{\partial U}{\partial \varphi} - \frac{V}{r^2}\Big), \\
\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{1}{r}\frac{\partial V}{\partial \varphi} = 0,
\end{gathered} \tag{1}$$

where p is the pressure, ρ is the density of the liquid, ν is the kinematic viscosity, and t is the time.

The solid-state rotation in the form of a tubular layer in the absence of gravitational forces is assumed to be the undisturbed motion of a liquid. Here the velocity and pressure components take on the following values: $U = 0$, $V = \omega r$, and $p = (1/2)\rho\omega^2(r^2 - c^2R^2)$.

The gravity causes stationary velocity and pressure perturbations in this steady-state motion. The velocity perturbations are considered insignificant compared with ωR, and the displacements of the free surface are assumed to be insignificant compared with cR. The problem is solved in the region near the free surface and far from the rigid wall. Therefore, the tangential stresses are not taken into account and the pressure is assumed to be determined only by gravity and the forces of inertia. After the replacement $\eta = r/R$ $(c \leqslant \eta \leqslant 1)$, one can approximate that

$$U = \omega R U_0, \quad V = \omega R(\eta + V_0), \quad p = (1/2)\rho\omega^2R^2[(\eta^2 - c^2) + p_0] - \rho g R\eta\cos\varphi, \tag{2}$$

where U_0, V_0, and p_0 are the stationary time-independent perturbations.

The boundary conditions on a rigid wall have the form

$$U_0 = V_0 = 0 \quad \text{for} \quad \eta = 1. \tag{3}$$

Let $\eta = c + \delta_0(\varphi)$ on the free surface with constant pressure, where δ_0 is the dimensionless displacement which is small compared with c. With allowance for (3), after transformations and equating the surface pressure to zero, the dynamic boundary condition on the free surface takes the form

$$p_0 + 2c\delta_0 = -2cg/(\omega^2 R)\cos\varphi \quad \text{for} \quad \eta = c. \tag{4}$$

The kinematic boundary condition on the free surface is as follows:

$$U_0 = \frac{\partial \delta_0}{\partial \varphi} \quad \text{for} \quad \eta = c. \tag{5}$$

The perturbations U_0 and V_0 are assumed to be small. Near the free surface and far from the rigid wall, it is possible to ignore the term $\partial^2 V/\partial r^2$. On the basis of (1), with allowance for (2) the equations of perturbed motion take the form

$$\begin{gathered}\frac{\partial U_0}{\partial \varphi} - 2V_0 = -\frac{1}{2}\frac{\partial p_0}{\partial \eta} + \frac{1}{\mathrm{Re_f}}\Big(\eta^2 \frac{\partial^2 U_0}{\partial \eta^2} + \frac{\partial^2 U_0}{\partial \varphi^2} + \eta\frac{\partial U_0}{\partial \eta} - 2\frac{\partial V_0}{\partial \varphi} - U_0\Big), \\ \frac{\partial V_0}{\partial \varphi} + 2U_0 = -\frac{1}{2\eta}\frac{\partial p_0}{\partial \varphi} + \frac{1}{\mathrm{Re_c}}\Big(\frac{\partial^2 V_0}{\partial \varphi^2} + 2\frac{\partial U_0}{\partial \varphi}\Big), \quad \frac{\partial}{\partial \eta}(\eta U_0) + \frac{\partial V_0}{\partial \varphi} = 0,\end{gathered} \tag{6}$$

where $\mathrm{Re_f} = \omega c^2 R^2/\nu$ is the Reynolds number on the free surface.

The boundary conditions on the rigid wall and the free surface have been formulated above [see (3)–(5)].

We now search for a solution in the form

$$\delta_0 = \Delta(\varphi)\cos\varphi, \quad p_0 = P(\eta,\varphi)\cos\varphi, \quad U_0 = \chi(\eta,\varphi)\sin\varphi, \quad V_0 = \xi(\eta,\varphi)\cos\varphi.$$

We now examine the approximate solution for $\varphi = \varphi^*$, where φ^* is the angle that corresponds to the site at which the tubular layer of a viscous liquid on the free surface begins to fail.

In the upper thickened part of the tubular layer, the gravitational and centrifugal forces are directed oppositely, and the necessary stability condition for the permanently perturbed motion of the fluid layer is a positive value of the radial pressure gradient [7]. According to numerical and experimental data [9, 10], the layer of a viscous liquid fails in the upper right part of its cross section for $\pi/2 < \varphi^* < \pi$ because of a certain lag of the free surface behind the rigid wall of the rotating cylinder. Based on the data of [10] and our results, for the angle φ^* with the greatest thickening of the tubular layer, which corresponds to the minimum radial pressure gradient on the free surface of a liquid, the dependence can be approximated in the form $\tan\varphi^* = -0.009c^{-1.5}$; note that $\varphi^* = \arctan(-0.009c^{-1.5}) + \pi$.

In the vicinity of φ^*, Eqs. (6) can be written in the form

$$\chi - 2\xi = -\frac{1}{2}p' + \frac{\tan\varphi^*}{\mathrm{Re_f}}(\eta^2\chi'' - 2\chi + \eta\chi' + 2\xi), \tag{7}$$

$$-\xi + 2\chi = \frac{1}{2\eta}P + \frac{1}{\tan\varphi^*\mathrm{Re_f}}(2\chi - \xi), \qquad \chi + \eta\chi' - \xi = 0$$

(the primes refer to the derivatives with respect to η), and the boundary conditions in the form $\chi = 0$ for $\eta = 1$; $P + 2c\Delta = -2c/\mathrm{Fr}$ and $\chi + \Delta = 0$ for $\eta = c$, where $\mathrm{Fr} = \omega^2 R/g$ is the Froude number on the cylinder surface.

After transformation of (7) and elimination of P, we obtain the equation for determining χ

$$\eta^2\chi'' + a\eta\chi' + b\chi = 0, \tag{8}$$

where $a = (3\mathrm{Re_f}\tan\varphi^* + 3\tan^2\varphi^* - 1)/(\mathrm{Re_f}\tan\varphi^* + \tan^2\varphi^* - 1)$ and $b = 1/(\mathrm{Re_f}\tan\varphi^* + \tan^2\varphi^* - 1)$. Solution (8) has the form

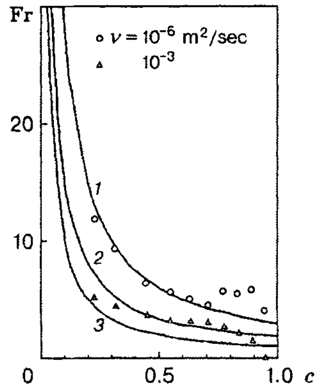

Fig. 2. Experimental check of the motion stability conditions for the tubular layer of a liquid for $R = 0.075$ m (points refer to the experiment, and curves to the calculation): 1) $\nu \to 0$ ($\mathrm{Fr} = 3/c$); 2) $\nu = 10^{-3}\mathrm{m}^2/\mathrm{sec}$; 3) $\nu \to \infty (\mathrm{Fr} = 1/c)$.

$$\chi = A(\eta^\alpha - \eta^\beta),$$

where $\alpha = -d - \sqrt{d^2 - b}$, $\beta = -d + \sqrt{d^2 - b}$, $d = (\mathrm{Re_f} \tan \varphi^* + \tan^2 \varphi^*)/(\mathrm{Re_f} \tan \varphi^* + \tan^2 \varphi^* - 1)$, and $A = -(1/\mathrm{Fr})\{[(\alpha - 1)c^\alpha - (\beta - 1)c^\beta][1/(\mathrm{Re_f} \tan \varphi^*) - 1] - (c^\alpha - c^\beta)\}^{-1}$. Therefore, we have $\xi(\eta, \varphi^*) = A[(\alpha+1)\eta^\alpha - (\beta-1)\eta^\beta]$, $P(\eta, \varphi^*) = 2A\eta[(\alpha-1)\eta^\alpha - (\beta-1)\eta^\beta][1/(\mathrm{Re_f} \tan \varphi^*) - 1]$, and $\Delta(\varphi^*) = -A(c^\alpha - c^\beta)$.

The stability condition for the stationary motion of the tubular layer of a liquid can be approximated in the form

$$\mathrm{Fr} > \frac{\cos \varphi^*}{f} [(\alpha - 1)\alpha c^\alpha - (\beta - 1)\beta c^\beta]\left(\frac{1}{\mathrm{Re_f} \tan \varphi^*} - 1\right)$$

$$\Big/\left\{[(\alpha - 1)c^\alpha - (\beta - 1)\beta c^\beta]\left(\frac{1}{\mathrm{Re_f} \tan \varphi^*} - 1\right) - (c^\alpha - c^\beta)\right\}. \quad (9)$$

As $\nu \to 0$ (ideal fluid), $\mathrm{Re_f} \to \infty$, and condition (9) degenerates to the condition obtained in [7]:

$$\mathrm{Fr} > 3/c. \quad (10)$$

As $\nu \to \infty$ (absolute viscous fluid), we have $\mathrm{Re_f} \to 0$, and condition (9) takes the form

$$\mathrm{Fr} > 1/c. \quad (11)$$

The experimental check of the stability conditions (9) is shown in Fig. 2 for water (circles) and castor oil (triangles). The steady-state modes above the curves correspond to the tubular form of motion, and those below the curves to the nontubular form. Figure 2 shows good agreement between the results of the solution of Eq. (9) and the experimental data.

The motion of a liquid (Fig. 3a) and a loose material (Fig. 3b) can be conventionally divided into a number of characteristic zones in the transverse cross section of the cylinder with qualitatively different flows. The combinations of the zones determine the modes of motion of a deformable material. For a loose material, the following characteristic modes connected with realization of technological processes [14] are distinguished in the order of increase in the velocity of rotation: the mode without throwing up (includes only the solid-state and falling-off zones), the mode with partial throwing up (the three zones appear), the mode with complete throwing up (no falling off occurs), the mode of partial centrifuging (on the surface of the cylinder, a solid-state tubular layer is formed from part of the material and this is followed by throwing up of the other part), and the tubular mode (the entire material forms a solid-state uniform tubular layer).

For quite a close similarity of the motion of a liquid and a loose deformable material in the cylinder as a whole (Fig. 3), the distinctive feature of the fluid motion is the effect of adhesion to a rigid wall and

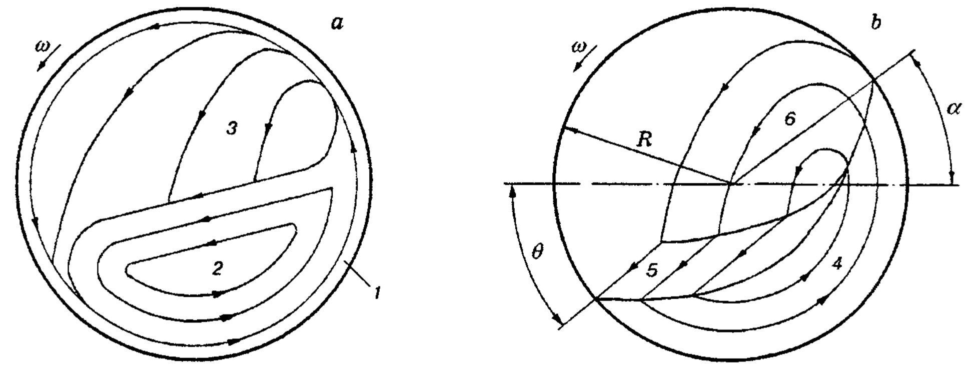

Fig. 3. Scheme of the characteristic motion zones in the cylinder for a liquid (a), where 1 is the tubular layer, 2 is the circulation zone, and 3 is the separation zone with falling, and for loose material (b), where 4 is the solid-state zone, 5 is the falling-off zone, and 6 is the throwing-up zone with falling.

coalescence together of the layers with relative slip, and the motion of a loose body is distinguished by the formation of a slope upon falling off and solid-state motion near the wall.

The experiments were performed on a setup equipped with nine changeable drums (R = 0.01325–0.21200 m) for a liquid and twelve drums (R = 0.0047–0.2120 m) for a loose material. For visualization of the motion, one face wall of the drums was transparent. As working liquids, water, spindle oil, and castor oil with $\nu = 10^{-6}$, $49 \cdot 10^{-6}$, and 10^{-3} m^2, respectively, were used. As working materials, three kinds of loose material with an average particle size of 0.5, 2, and 10 mm, respectively, were used. The degree of filling of the drum hollow with the material $æ$ ranged from 0.10 to 0.95 [$æ = \tau/(\pi R^2 L)$], where τ is the volume of the material in the cylinder hollow and L is the length of the hollow.

The velocities of stationary rotation of the drum upon formation and fracture of the tubular layer during steady motion followed by smooth acceleration and deceleration were measured. For a loose material, the velocities of rotation corresponding to transitions of the characteristic modes of motion were determined. For $æ = 0.1$, the experimental data for a liquid are close to the results of [3, 4, 10].

An analysis of the experimental results showed that, in this case, the similarity criteria of steady-state motion of a deformable material are the Reynolds and Froude (Re and Fr) numbers on the radial surface of the cylinder and the degree of filling of the cylinder with material $æ$. The first criterion characterizes friction forces, the second criterion characterizes the forces of inertia, and the third one characterizes the geometrical motion parameters. The experiments showed that the frictional properties upon motion of all the loose materials examined [the angle of natural slope in the motion is $\theta \approx 30°$ (Fig. 3b) and the angular velocities of the cylinder, which correspond to transitions of the modes] were almost identical. Since all the loose materials in the motion considered can be regarded as liquids with quite close values of viscosity, it is assumed that $\nu \approx 1$ m^2/sec for simplicity; as the experiment shows, this is close to the value of ν for a liquid with a similar behavior under similar conditions.

Figure 4 shows a comparative graphic analysis of the motion stability conditions for a liquid layer according to [7], our analytical results obtained by means of (9), and the experimental data. In the coordinates of Re and Fr, for $æ = 0.5$ the curves that correspond to transition of the tubular to the nontubular mode and inversely are constructed. The zone above the boundary corresponds to the tubular form of motion. The sloping dashed straight lines correspond to the modes of fluid motion in a constant-radius cylinder rotating with different velocities. For large Re, the phenomenon of mode hysteresis is observed, i.e., the velocity of rotation of the cylinder upon formation of a tubular layer during its acceleration exceeds the rate of layer fracture upon deceleration [3, 4, 10]. For small Re, the rates of formation of a layer during acceleration and the rates of fracture during deceleration are the same; this is due to the appearance of secondary circulating flows in the form of a cylinder on the internal surface of the layer [10]. The resulting condition (9) agrees

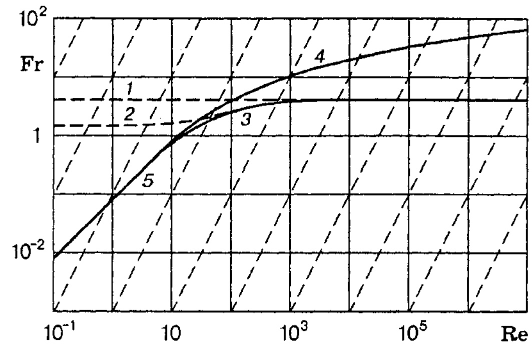

Fig. 4. Diagram of transition of the tubular mode of motion of a liquid to the nontubular mode and inversely for æ = 0.5: dashed curves 1 and 2 are the data calculated according to [7] and (9), respectively, solid curves are the experimental data corresponding to transition of the tubular to the nontubular mode upon deceleration of the cylinder (3), the nontubular to the tubular one upon acceleration (4), and the tubular to the nontubular mode and inversely upon deceleration or acceleration (5).

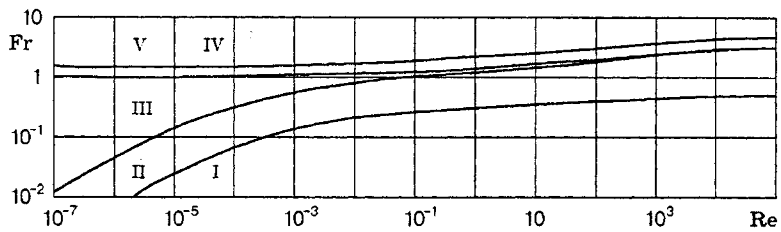

Fig. 5. Diagram of transition of the modes of motion of a loose material for æ = 0.5: curve I refers to the throwing-up-free mode, II to the mode with partial throwing up, III to the mode with complete throwing up, IV to the mode with partial centrifuging, and V to the tubular mode.

well with the experimental data for the angular rate of fracture of the layer upon deceleration of the cylinder; however, it becomes incorrect when secondary flows emerge.

Figure 5 shows the diagram of transition of the modes of motion of a loose material in the cylinder for æ = 0.5; this diagram was constructed with the use of the experimental data obtained in the coordinates of Re and Fr. The hysteresis for the loose medium was not observed. As Re $\to \infty$, the ordinate of the boundary of modes IV and V corresponds to the condition Fr $\to 3/c$ on the free surface of the layer; this condition corresponds to (10), and the ordinate of the boundary of modes III and IV to a similar condition Fr $\to 3$ on the cylinder surface. As Re $\to 0$, the ordinate of the boundary of modes IV and V corresponds to the condition Fr $\to 1/c$ on the free surface which corresponds to (11), and the ordinate of the boundary of modes III and IV to a similar condition Fr $\to 1$ on the cylinder surface. In addition, as Re $\to \infty$, the boundary of modes III and IV merges with the boundary of modes IV and V, and the ordinate of the boundary of modes I and II corresponds to the condition Fr $\to \sin\alpha$ (Fig. 3b), which corresponds to particle separation from the cylinder surface [14]. As Re $\to 0$, the ordinates of the boundaries of modes I–III tend to zero.

To determine the velocity of rotation of the cylinder ω that corresponds to transition of the modes of motion of a deformable material by means of the diagrams, a sloping straight line is constructed with the use of known parameters R and ν for a current velocity, as done in Fig. 4. The quantity ω is calculated with the use of the coordinates of the intersection point of this straight line with the diagram for a corresponding degree of filling.

A comparative analysis of the diagrams shows that the motion stability condition in a rotating horizontal cylinder of the tubular layer of a liquid and a loose medium as varieties of deformable material are identical. The onset of the formation of a layer of a loose material on the cylinder surface is also a similar condition. These conditions depend greatly on the values of Re and æ; therefore, the "critical" velocity of rotation of the cylinder [14], which corresponds to the condition Fr $\to$ 1 on the free surface and is used to calculate technological modes, determines inadequately the boundaries of transition of the tubular and nontubular forms of motion of a material. The resulting diagrams can be used for more correct calculations.

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